ENHANCED BINDING IN NON-RELATIVISTIC QED

CHRISTIAN HAINZL\(^1\), VITALI VOGALTER, AND SEMJON A. VUGALTER

ABSTRACT. We consider a spinless particle coupled to a photon field and prove that even if the Schrödinger operator \( p^2 + V \) does not have eigenvalues the system can have a ground state. We describe the coupling by means of the Pauli-Fierz Hamiltonian and our result holds in the case where the coupling constant \( \alpha \) is small.

1. INTRODUCTION

In the picture of Quantum electrodynamics (QED) atoms consist of charged particles, which are necessarily coupled to a photon field. If one neglects the radiation effects one obtains the standard Schrödinger operator. Although the fundamental properties of the one-particle and multi-particle Schrödinger operators have been successfully studied since the middle of the last century, the systematic mathematical study of the non-relativistic QED model was initiated by Bach, Fröhlich, and Sigal in [BFS1, BFS2, BFS3] only a couple of years ago (a comprehensive review of results in non-relativistic QED can be found in [GLL]) and some very fundamental problems remain still open. One of these problems is the question of enhanced binding via interaction with a quantized radiation field.

Consider a particle in a potential well \( \beta V(x) \) with \( V(x) \leq 0 \). If the potential well is not deep enough, i.e. \( \beta \) is small, the corresponding Schrödinger operator does not have a discrete spectrum and binding does not occur. There exists a critical value \( \beta_0 \) such that for \( \beta > \beta_0 \) there is at least one bound state whereas for \( \beta \leq \beta_0 \) no particle can be bound.

In a recent paper Griesemer, Lieb, and Loss ([GLL]) proved that a photon field cannot decrease the binding energy. If the Schrödinger operator with potential \( \beta V \) has an eigenvalue, the corresponding energy operator in non-relativistic QED (Pauli-Fierz Hamiltonian) has a ground state.

However, the physical intuition tells us that interaction with a photon field must increase binding. According to the photon cloud surrounding the particle, the effective mass of the electron increases and consequently it needs more energy to leave the potential well.

The goal of this paper is to give a mathematical rigorous proof this phenomenon. Previously the enhanced binding was studied in the dipole approximation by Hiroshima and Spohn in [HS]. In this approximation it is

\(^{1}\text{Marie Curie Fellow}\)

\( Date: \) December 17, 2002.
assumed that the magnetic vector potential does not depend on the coordinates of the particle. They proved that, if the potential \( \beta V \) is fixed, for sufficiently large values of the coupling parameter \( \alpha \) (which is the fine structure constant, see (1)), binding takes place.

Our approach to this problem is different. On the one hand we study the Pauli-Fierz operator without additional restrictions on the magnetic vector potential and on the other hand our results hold for small values of \( \alpha \) (recall, that the physical value of \( \alpha \) is about \( 1/137 \)). We prove that in case of the Pauli-Fierz operator (and for small \( \alpha \) enough) binding starts at values of \( \beta \) strictly less than \( \beta_0 \).

2. MAIN RESULTS

We describe the self-energy of the particle by

\[
T = (p + \sqrt{\alpha A(x)})^2 + H_I,
\]

acting on the Hilbert space

\[
\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F},
\]

where \( \mathcal{F} \) is the symmetric Fock space for the photon field.

We use units such that \( \hbar = c = 1 \) and the mass \( m = \frac{1}{2} \). The electron charge is then given by \( e = \sqrt{\alpha} \), with \( \alpha \approx 1/137 \) the fine structure constant. In the present paper \( \alpha \) plays the role of a small, dimensionless number. Our results hold for sufficiently small values of \( \alpha \). The electron momentum operator is \( p = -i \nabla_x \), while \( A \) is the magnetic vector potential. We fix the Coulomb gauge \( \text{div} A = 0 \).

The vector potential is

\[
A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{2\pi |k|^{1/2}} \varepsilon_{\lambda} [a_{\lambda}(k)e^{ikx} + a_{\lambda}^*(k)e^{-ikx}] dk,
\]

where the operators \( a_{\lambda}, a_{\lambda}^* \) satisfy the usual commutation relations

\[
[a_{\nu}(k), a_{\lambda}^*(q)] = \delta(k - q)\delta_{\lambda,\nu}, \quad [a_{\lambda}(k), a_{\nu}(q)] = 0.
\]

The vectors \( \varepsilon_{\lambda}(k) \in \mathbb{R}^3 \) are two orthonormal polarization vectors perpendicular to \( k \).

Obviously,

\[
A(x) = D(x) + D^*(x),
\]

where

\[
D(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{2\pi |k|^{1/2}} \varepsilon_{\lambda} a_{\lambda}(k)e^{ikx} dk = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} G_{\lambda}(k)a_{\lambda}(k)e^{ikx} dk,
\]

and \( D^* \) is the operator adjoint to \( D \).

The function \( \chi(|k|) \) describes the ultraviolet cutoff for the interaction at large wavenumbers \( k \). For convenience we choose \( \chi \) to be the Heaviside function \( \Theta(A - |k|/l_C) \), where \( l_C = \hbar/(mc) \) is the Compton wavelength. In our units \( l_C = 2 \). Our proof would work for any other cut-off.
The photon field energy $H_f$ is given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \vert k \vert a_\lambda^*(k) a_\lambda(k) dk,$$  

(7)

whereas

$$P_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \vert k \vert a_\lambda^*(k) a_\lambda(k) dk$$

(8)

denotes the field momentum.

To prove existence of enhanced binding in non-relativistic QED we would like to compare binding in the presence of the photon field and without it. To this end let us introduce the Schrödinger operator

$$h_\beta = -\Delta + \beta V(x)$$

(9)

with external potential $\beta V(x) \in C(\mathbb{R}^3)$, which we assume to be radial $V(x) = V(|x|)$, non-positive, $V(x) \leq 0$, and with compact support. It is known that there is a critical value of the parameter $\beta_0 > 0$ such that for $\beta \leq \beta_0$ there is no ground state and the operator (9) has only an essential spectrum and at the same time for all $\beta > \beta_0$ the operator $h_\beta$ has at least one eigenvalue.

The corresponding operator with a quantized radiation field is

$$\mathbf{H}_\beta = T + \beta V(x).$$

(10)

[Hi] guarantees the self-adjointness of $\mathbf{H}_\beta$ on the domain $\mathcal{D}(p^2 + H_f)$. Our goal is to show that the operator $\mathbf{H}_\beta$ has a bound state for values of $\beta$ strictly smaller than $\beta_0$. To establish the existence of a ground state of $\mathbf{H}_\beta$ we apply the criterion of [GLL], which says that $\mathbf{H}_\beta$ has a ground state if

$$\inf \text{spec } \mathbf{H}_\beta < \inf \text{spec } T.$$  

(11)

However, in contrast to the Schrödinger operator $h_\beta$, for which the infimum of the spectrum without potential is always 0, the inf spec $T$ is a complicated function depending on $\alpha$ and $\Lambda$. To prove the inequality (11) one needs precise estimates on this function. Our first result is the following asymptotic estimate on $\Sigma_\alpha = \inf \text{spec } T$.  

**THEOREM 1** (Localization of the spectrum of a free spinless particle). Let

$$\mathcal{E}_0 = \langle 0 | D \cdot D [P_f^2 + H_f]^{-1} D^* \cdot D^* | 0 \rangle,$$  

(12)

with $D^* = D^*(0)$ and $|0\rangle$ denotes the vacuum of $\mathcal{F}$. Then, for small $\alpha$,

$$\left| \Sigma_\alpha - \alpha \pi^{-1} \Lambda^2 + \alpha^2 \mathcal{E}_0 \right| \leq C \alpha^3 \Lambda^4,$$  

(13)

where $C > 0$ is an appropriate constant independent of $\alpha$ and $\Lambda$.  

**REMARK 1.** The number $\mathcal{E}_0$ can be computed directly through the integral

$$\mathcal{E}_0 = \sum_{\mu, \nu=1,2} 2 \int \frac{[G^\mu(k_1) \cdot G^\nu(k_2)]^2}{|k_1 + k_2|^2 + |k_1| + |k_2|} dk_1 dk_2.$$  

(14)
The first to leading order is obtained by perturbation theory. One of our main goals here is to prove that perturbation theory is correct in the case when $\Lambda$ is fix, which is a non-trivial problem, since there is no isolated eigenvalue and Kato's perturbation methods cannot be applied.

Recall, that the operator $h_{\beta}$ has a critical value $\beta_0$ of the parameter $\beta$ such that, for $\beta \leq \beta_0$, $h_{\beta}$ does not have a bound state. Using Theorem 1 we construct a variational trial function proving for small values of $\alpha$ the following:

**THEOREM 2** (Enhanced binding). *For all sufficiently small $\alpha$ there exists a number $\beta_1(\alpha) < \beta_0$, such that for all $\beta > \beta_1(\alpha)$ the operator $H_{\beta}$ has a ground state.*

Observe that the converse statement is not proven, namely we do not obtain a $\beta_2(\alpha) > 0$ such that for $\beta < \beta_2(\alpha)$ the ground state does not exist.

**REMARK 2.** Concerning the critical case $\beta = \beta_0$ the proof of Theorem 2 in particular implies that there exists a real number $\rho > 0$ such that $H_{\beta_0}$ has a ground state for all $\alpha \in (0, \rho]$.

Of course we expect that binding holds on, or even increases, when $\alpha$ gets large, but we cannot prove it due to the fact that we can only control the self-energy for small $\alpha$.

### 3. Proof of Theorem 1

Let us start with a free spinless electron. In this case the Hamiltonian is translation invariant, which means that it commutes with the total momentum $p + P_f$. It is therefore possible to rewrite the Hilbert space and the Hamiltonian as a direct integral

\[
\mathcal{H} = \int_{\mathbb{R}^3} d^3 p \mathcal{H}_p
\]

and

\[
T = \int_{\mathbb{R}^3} d^3 p T_P,
\]

with $T_P$ acting on $\mathcal{H}_P$. Each $\mathcal{H}_P$ is isomorphic to $\mathcal{F}$. In this representation $T_P$ is given by

\[
T_P = (P - P_f + \sqrt{\alpha} A(0))^2 + H_f.
\]

According to [F] the minimum of $\inf \spec T_P$ is achieved for $P = 0$, which tells us that we only need to consider the operator

\[
T_0 = (P_f + \sqrt{\alpha} A)^2 + H_f.
\]

Throughout this section we use $A = A(0), D = D(0)$, and $D^* = D^*(0)$. We define $\Sigma_\alpha = \inf \spec T_0$.

It turns out to be convenient to denote a general $\Psi \in \mathcal{H}$ as

\[
\Psi = \{\psi_0, \psi_1, \ldots, \psi_n, \ldots\},
\]
where
\[ \psi_n = \psi_n(x, k_1, \ldots, k_n; \lambda_1, \ldots, \lambda_n). \] (20)

In order not to overburden the paper with too many indices we will suppress the photon variables in \( \psi_n \), when it does not lead to misunderstanding.

3.1. Upper bound. We take the trial state
\[ \Psi = \{ |0\rangle, 0, -\alpha[P_f^2 + H_f]^{-1} D^* \cdot D^*|0\rangle, 0, 0, \ldots \} \] (21)
where \(|0\rangle \in \mathbb{C}, \langle 0|0 \rangle = 1\), denotes the vacuum vector. The photon part of \( \Psi \) can be written explicitly as
\[ -\sum_{\lambda, \mu = 1, 2} \alpha \sqrt{2} \frac{1}{(k_1 + k_2)^2 + |k_1|^2 + |k_2|^2} G^\lambda(k_1) \cdot G^{\mu}(k_2)|0\rangle. \] (22)
Since
\[ D \cdot D^* - D^* \cdot D = \pi^{-1} A^2, \] (23)
obviously
\[ A^2 = \pi^{-1} A^2 + 2D^* \cdot D + D \cdot D + D^* \cdot D^*. \] (24)
Therefore, since \( \|\Psi\|^2 \geq 1 \),
\[ (\Psi, T_0 \Psi)/(\Psi, \Psi) \leq \alpha \pi^{-1} A^2 - \alpha^2 \langle 0|D \cdot D [P_f^2 + H_f]^{-1} D^* \cdot D^*|0\rangle \]
\[ + 2\alpha^2 \|D [P_f^2 + H_f]^{-1} D^* \cdot D^*|0\rangle \|^2. \] (25)
One can easily see by scaling that the last two terms in the r.h.s. of (25) are of order \( \Lambda^2 \).

3.2. Lower bound\(^1\). We start with some a priori estimates.

**LEMMA 1.**
\[ T_0 \geq \alpha \pi^{-1} A^2 - \text{const.} \alpha^2 \Lambda^3 + \frac{1}{2}(P_f^2 + H_f). \] (26)

**Proof.** Since \([P_f, A] = 0\)
\[ T_0 = P_f^2 + 2\sqrt{\alpha} P_f \cdot A + \alpha A^2 + H_f. \] (27)
By means of Schwarz’s inequality
\[ 2\sqrt{\alpha} P_f \cdot A = 4\sqrt{\alpha} \text{Re}(P_f \cdot D) \leq \frac{1}{2} P_f^2 + 8\alpha D^* D \] (28)
and
\[ \alpha(D \cdot D + D^* \cdot D^*) \leq C^{-1} D^* \cdot D + \alpha^2 C D \cdot D^* \] (29)
for any \( C > 0 \). Using (24) we obtain
\[ T_0 \geq (\pi^{-1} \alpha A^2 - C \pi \alpha^2 \Lambda^2) + H_f \left( \frac{1}{2} - \frac{8\alpha \Lambda}{\pi} - C^{-1} \frac{2}{\pi} - \frac{\alpha^2 C}{\pi} \Lambda \right) \]
\[ + \frac{1}{2}(P_f^2 + H_f), \] (30)
\(^1\)A different proof of the lower bound, based on partitions of unity of the photon configuration space and improved estimates for the localization errors for the relativistic energy, can be found in the preprint version [HVV].
which implies the lemma with \( C = \bar{c} \Lambda \), with an appropriate \( \bar{c} > 0 \), and \( \alpha \Lambda \) and \( \alpha \) not too large. \( \square \)

**Remark 3.** We know from the upper bound that any approximate ground state \( \Psi_0 \) satisfies (\( \Psi_0, T_0 \Psi_0 \leq \alpha \pi^{-1} \Lambda^2 + O(\alpha^2) \)). Therefore by Lemma 1 we infer the a priori estimate

\[
(\Psi_0, [P_f^2 + H_f] \Psi_0) \leq \text{const.} \alpha^2 \Lambda^3. \tag{31}
\]

Using (24) we derive

\[
(\Psi_0, T_0 \Psi_0) \geq \alpha \pi^{-1} \Lambda^2 \| \Psi_0 \|^2 + \sum_{n=0}^{\infty} \mathcal{E}[\psi_n, \psi_{n+1}, \psi_{n+2}], \tag{32}
\]

where

\[
\mathcal{E}[\psi_n, \psi_{n+1}, \psi_{n+2}] = (\psi_{n+2}, \mathcal{A} \psi_{n+2}) + 2\Re \left( [2\sqrt{\alpha} P_f \cdot D^* \psi_{n+1} + \alpha D^* \cdot D^* \psi_n, \psi_{n+2}] \right), \tag{33}
\]

with

\[
\mathcal{A} = P_f^2 + H_f. \tag{34}
\]

Recall, in our notation

\[
\Psi_0 = \{ \psi_0, \psi_1(k_1), \ldots, \psi_n(k_1, \ldots, k_n), \ldots \}. \tag{35}
\]

We consider the term \( \mathcal{E}[\psi_n, \psi_{n+1}, \psi_{n+2}] \). If we set

\[
f = A^{1/2} \psi_{n+2}, \quad g = -A^{-1/2} \left[ \sqrt{\alpha} 2 P_f \cdot D^* \psi_{n+1} + \alpha D \cdot D^* \psi_n \right], \tag{36}
\]

then by means of \( \| f \|^2 - 2 \Re (f, g) \geq -\| g \|^2 \) we derive

\[
\mathcal{E}[\psi_n, \psi_{n+1}, \psi_{n+2}] \geq -\left\| 2\sqrt{\alpha} A^{-1/2} P_f \cdot D^* \psi_{n+1} + \alpha A^{-1/2} D \cdot D^* \psi_n \right\|^2. \tag{37}
\]

Let us start the estimation of the r.h.s. of (37) with

\[
-\alpha^2 (\psi_n, D \cdot D [P_f^2 + H_f]^{-1} D^* \cdot D^* \psi_n). \tag{38}
\]

It will be shown that it produces the first to leading order in \( \alpha^2 \). Recall,

\[
[D^* \cdot D^* \psi_n]_{n+2} = \frac{1}{(n+2)(n+1)} \sum_{\lambda, \mu=1,2} \sum_{j=1}^{n+2} \sum_{j \neq k} G^\mu(k_j) \cdot G^\lambda(k_i) \times

\times \psi_n(k_1, \ldots, k_j, \ldots, k_i, \ldots, k_{n+2}), \tag{39}
\]

where \( k_j \) indicates that the \( j \)-th variable is omitted. By permutational symmetry we can distinguish between three different terms,

\[
(\psi_n, D \cdot D [P_f^2 + H_f]^{-1} D^* \cdot D^* \psi_n) = I_n + II_n + III_n, \tag{40}
\]

which come out quite naturally when we insert equation (39) into (40) and have in mind that the l.h.s. of (40) can be written as

\[
(D^* \cdot D^* \psi_n, [P_f^2 + H_f]^{-1} D^* \cdot D^* \psi_n). \tag{41}
\]
First, the diagonal part $I_n$ appears, when in the right hand side of (41) as well as in the left hand side two photons $G^\mu(k_j) \cdot G^\lambda(k_i)$ with the same variables $k_i, k_j$ are produced,

$$I_n = \sum_{\lambda, \mu = 1, 2} 2 \int \left[ \frac{|G^\lambda(k_1) \cdot G^\mu(k_2)|^2 |\psi_n(k_3, \ldots, k_{n+2})|^2}{\sum_{i=1}^{n+2} k_i^2 + \sum_{i=1}^{n+2} |k_i|} dk_1 \ldots dk_{n+2}. \right. \tag{42}$$

If we set $Q = |\sum_{i=3}^{n+2} k_i^2 + |k_1 + k_2|^2 + \sum_{i=1}^{n+2} |k_i|$ and $b = 2 |\sum_{i=3}^{n+2} k_i| \cdot |k_1 + k_2|$ and use the expansion

$$\frac{1}{Q + b} = \frac{1}{Q} - \frac{1}{Q} \frac{b}{Q} \frac{1}{Q + b}$$

then we see that the second term vanishes when integrating over $k_1, k_2$. Therefore, with $Q \geq |k_1 + k_2|^2 + |k_1| + |k_2|$ and $Q + b \geq |k_1| + |k_2|$ we arrive at

$$I_n \leq \sum_{\lambda, \mu = 1, 2} 2 \left[ \|\psi_n\|^2 \int \frac{|G^\lambda(k_1) \cdot G^\mu(k_2)|^2}{|k_1 + k_2|^2 + |k_1| + |k_2|} dk_1 dk_2 \right.$$  

$$+ 4 \int \frac{|G^\lambda(k_1)|^2 |G^\mu(k_2)|^2 |k_1 + k_2|^2}{|k_1 + k_2|^2 + |k_1| + |k_2|} \times$$  

$$\times \left| \sum_{i=3}^{n+2} k_i \right|^2 |\psi_n(k_3, \ldots, k_{n+2})|^2 dk_1 \ldots dk_{n+2} \right]$$

$$\leq \langle 0 | D \cdot D | P_f^2 + H_f \rangle^{-1} D^* \cdot D^* \langle 0 | \psi_n \rangle^2 + \text{const.} A \langle P_f \psi_n \rangle^2. \tag{44}$$

For convenience we define the operator $|D|$ by

$$|D| = \sum_{\lambda, \mu = 1, 2} \int |G^\lambda(k)| a_\lambda(k) dk. \tag{45}$$

$|D|^*$ denotes the operator adjoint. Obviously, $|D|$, Lemma A. 4 still holds for $|D|$, namely

$$|D|^* |D| \leq \frac{2}{\pi} H_f. \tag{46}$$

The second term $II_n$ occurs, when a term $G^\mu(k_j) \cdot G^\lambda(k_i)$ in the l.h.s. of (41) meets a two photon part $G^\mu(k_j) \cdot G^\lambda(k_i)$ in the r.h.s. of (41). Using $P_f^2 + H_f \geq H_f$ we evaluate

$$II_n \leq (n + 1) \sum_{\lambda, \mu = 1, 2} \int \frac{|G^\lambda(k_1)| |G^\mu(k_2)| |G^\lambda(k_1)| |G^\mu(k_{n+2})|}{\sum_{i=1}^{n+2} k_i}$$

$$\times |\psi_n(k_3, \ldots, k_{n+2})||\psi_n(k_2, \ldots, k_{n+1})| dk_1 \ldots dk_{n+2}$$

$$\leq \text{const.} \int \frac{|G(k_1)|^2}{|k_1|} dk_1 \langle \psi_n, |D|^* D |\psi_n \rangle \leq \text{const.} \Lambda^2(\psi_n, H_f \psi_n). \tag{47}$$
Finally, the third term, where the indices of produced photons in the right
hand side differ completely from the indices in the left hand side of (41),
can be bounded by

\[ III_n \leq (n + 1)^2 \sum_{\lambda, \mu = 1, 2} \int \frac{|G^\lambda(k_1)| |G^\mu(k_2)| |G^\lambda(k_{n+1})| |G^\mu(k_{n+2})|}{\sum_{i=1}^{n+2} |k_i|} \times \]

\[ \times |\psi_n(k_3, \ldots, k_{n+2})||\psi(k_1, \ldots, k_n)|dk_1 \ldots dk_{n+2} \]

\[ \leq \text{const.}(\psi_n, |D^*H_f^{-1/2}|D|D^*H_f^{-1/2}|D|\psi_n) \leq \text{const.} \Lambda(\psi_n, H_f|\psi_n), \quad (48) \]

where we used

\[ \sum_{i=1}^{n+2} |k_i| \geq \left( \sum_{i=1}^{n+1} |k_i| \right)^{1/2} \sum_{i=2}^{n+2} |k_i| \]

(49)

the fact that we can write

\[ [H_f]^{-1/2}\psi_n(k_1, \ldots, k_n) = \left( \sum_{i=1}^{n} |k_i| \right)^{-1/2} \psi_n(k_1, \ldots, k_n), \quad (50) \]

and (46).

We summarize

\[ -\alpha^2 (\psi_n, D \cdot DA^{-1} D^* \cdot D^* \psi_n) \geq -\alpha^2 (|D|D[P_f^2 + H_f]^{-1}D^* \cdot D^*|0\rangle \langle 0| \psi_n \|^2 \]

\[ - \text{const.} \Lambda(\|P_f \psi_n\|^2 + \Lambda(\psi_n, H_f|\psi_n)). \quad (51) \]

The second diagonal term of (37) reads

\[ - \alpha(P_f \cdot D^* \psi_{n+1}, A^{-1}P_f \cdot D^* \psi_{n+1}) = \sum_{\lambda=1, 2} -\alpha \times \]

\[ \times \left[ \int \frac{[G^\lambda(k_{n+2}) \cdot (\sum_{i=1}^{n+1} k_i)]^2 |\psi_{n+1}(k_1, \ldots, k_{n+1})|}{\sum_{i=1}^{n+2} k_i + \sum_{i=1}^{n+2} |k_i|} dk_1 \ldots dk_{n+2} \]

\[ + (n + 1) \int \frac{[G^\lambda(k_1) \cdot (\sum_{i=1}^{n+2} k_i)] [G^\lambda(k_{n+2}) \cdot (\sum_{i=1}^{n+2} k_i)]}{\sum_{i=1}^{n+2} k_i + \sum_{i=2}^{n+2} |k_i|} \times \]

\[ \times |\psi_{n+1}(k_1, \ldots, k_{n+2})| \psi_{n+1}(k_2, \ldots, k_{n+2}) dk_1 \ldots dk_{n+2} \]

\[ \geq -\text{const.} \alpha \left( \Lambda \|P_f \psi_{n+1}\|^2 + \langle \psi_{n+1}, |D^*|D|\psi_{n+1}\rangle \right). \quad (52) \]

For the second term in the r.h.s. we used first

\[ \frac{\left| \sum_{i=1}^{n+2} k_i \right|^2}{\sum_{i=1}^{n+2} k_i + \sum_{i=2}^{n+2} |k_i|} \leq 1. \quad (53) \]
By (46) and Schwarz’s inequality for the off-diagonal term in (37), as well as summing over all $n$ and using the a priori knowledge (31) we arrive at the desired result.

4. PROOF OF THEOREM 2

To prove the Theorem we will check the binding condition of [GLL] for $\beta = \beta_0$. Namely, we will show that

$$\inf \text{spec } \mathbf{H}_{\beta_0} < \Sigma_\alpha - \delta \alpha^2 + \mathcal{O}(\alpha^{5/2}).$$

The binding for all $\beta \in (\beta_1, \beta_0]$ with some $\beta_1 < \beta_0$ follows from (54) and the continuity of the quadratic form in $\beta$. In the proof of Theorem 1 we have seen that the trial state

$$\Psi_n = \{|0\rangle, 0, \alpha [P_f^2 + H_f]^{-1} D(0)^* \cdot D(0) \cdot |0\rangle, 0, 0, \ldots\},$$

recovers the self energy up to the order $\alpha^2$. Our next goal is to modify this trial state in such a way that for the modified state $\Psi^0 \in \mathcal{H}$

$$(\Psi^0, \mathbf{H}_{\beta_0} \Psi^0) \leq (\Sigma_\alpha - \delta \alpha^2 + \mathcal{O}(\alpha^{5/2})) \| \Psi \|^2,$$

with some $\delta > 0$.

Throughout the previous section we worked with the operator $A(0)$. Here, our Hamiltonian depends on the electron variable $x$. In order to adapt our methods developed in the previous section we introduce the unitary transform

$$U = e^{i P_f \cdot x}$$

acting on the Hilbert space $\mathcal{H}$. Applied to a $n$-photon function $\varphi_n$ we obtain $U \varphi_n = e^{i (\sum_{i=1}^n k_i) \cdot x} \varphi_n(x, k_1, \ldots, k_n)$ and additionally $U(D^*(x) \psi(x)) = G(k) \psi(x)$.

Since $U p U^* = p - P_f$ we infer for our Hamiltonian $\mathbf{H}_{\beta_0}$

$$U \mathbf{H}_{\beta_0} U^* = (p - P_f + \sqrt{\alpha} A)^2 + H_f + \beta_0 V(x) \equiv \tilde{\mathbf{H}}_{\beta_0},$$

with $A = A(0)$. Obviously,

$$\inf \text{spec } \tilde{\mathbf{H}}_{\beta_0} = \inf \text{spec } \mathbf{H}_{\beta_0}.$$  

Therefore, for convenience, we will work in the following with the operator $\tilde{\mathbf{H}}_{\beta_0}$.

Next, we define our trial function

$$\Psi^0 = \{f, -d \sqrt{\alpha} A^{-1} p \cdot D^* f, -\alpha A^{-1} D^* \cdot D^* f, 0, 0, \ldots\},$$

with $A = P_f^2 + H_f$, $D = D(0)$, and $d$ an appropriate constant which will be chosen later.

We assume $f(x) \in C_0^2(\mathbb{R}^3)$ to be a real, spherically symmetric function and to fulfill the condition

$$\|p^2 f(x)\| \leq C_1 \|p f(x)\| \leq C_2 \sqrt{\alpha} \|f(x)\|,$$

with some constants $C_{1,2}$. 

For short, denote the 1- and 2-photon terms in $\Psi^0$ as $\psi_1$ respectively $\psi_2$. Obviously, the terms $(\psi_1, P_f \cdot p \psi_1)$ and $(\psi_2, P_f \cdot p \psi_2)$ vanish. This can be seen by integrating over the field variables having in mind that the reflection $k \rightarrow -k$ commutes with $\mathcal{A}$.

By means of (61) and Schwarz’s inequality we obtain
\[
2\sqrt{\alpha(p \cdot D^* \psi_1, \psi_2)} + |(\psi_2, p^2 \psi_2)| \leq \|\Psi^0\|^2 \mathcal{O}(\alpha^{5/2}).
\]  
(62)

We now use our knowledge from the proof of Theorem 1 to obtain
\[
\alpha^{-1}A^2 \|\Psi^0\|^2 + (\psi_2, [P^2_f + H_f] \psi_2) + 2\alpha \Re(D^* \cdot D^* \psi_2) = \\
= [\Sigma_\alpha + \mathcal{O}(\alpha^3)] \|\Psi^0\|^2. \tag{63}
\]

Taking into account that $V \leq 0$ we arrive at
\[
(\Psi^0, \mathbf{H}_\beta \Psi^0) \leq (f, [p^2 + \beta_0 V] f) - d\alpha(f, p \cdot D A^{-1} p \cdot D^* f) + \\
+ \alpha d^2 \left[(f, p \cdot D A^{-1} p \cdot D^* f) + (f, p \cdot D A^{-1} p \cdot D^* f)\right] + \\
+ [\Sigma_\alpha + \mathcal{O}(\alpha^{5/2})] \|f\|^2. \tag{64}
\]

Using the Fourier transform we are able to evaluate explicitly
\[
(f, p \cdot D A^{-1} p \cdot D^* f) = \sum_{\lambda = 1, 2} \int \left|\hat{f}(t)\right|^2 \frac{|G^\lambda(k) \cdot t|^2}{|k|} dk dt = C_3 \|pf\|^2 \tag{65}
\]
and additionally get
\[
(f, p \cdot D A^{-1} p^2 A^{-1} p \cdot D^* f) = C_3 \|p^2 f\|^2 \leq C_4 \|pf\|^2, \tag{66}
\]
where we used (61). This implies
\[
(\Psi^0, \mathbf{H}_\beta \Psi^0) \leq \left(1 - C_3\alpha d + d^2\alpha(C_3 + C_4)\right)(f, p^2 f) + (f, \beta_0 V f) + \\
+ [\Sigma_\alpha + \mathcal{O}(\alpha^{5/2})] \|\Psi^0\|^2. \tag{67}
\]

As the next step we choose $d < \frac{C_3}{2(C_3 + C_4)}$ which gives
\[
(\Psi^0, \mathbf{H}_\beta \Psi^0) \leq (1 - \nu\alpha) \|pf\|^2 + \beta_0(f, V f) + [\Sigma_\alpha + \mathcal{O}(\alpha^{5/2})] \|\Psi^0\|^2, \tag{68}
\]
where
\[
\nu = \frac{C_3^2}{4(C_3 + C_4)}. \tag{69}
\]

Due to our choice of $\beta_0$, obviously the operator
\[
-(1 - \nu\alpha) \Delta + \beta_0 V(x) \tag{70}
\]
has at least one negative eigenvalue. However, the r.h.s. of (68) contains the terms which are of order $\mathcal{O}(\alpha^{5/2})$ and to prove Theorem 2 we have to provide more precise estimates on the negative eigenvalues of (70). The required estimate is given by Lemma 2 in the Appendix.

Applying this Lemma with $\nu\alpha = \gamma$ completes the proof of the theorem.
Appendix A. Auxiliary Lemma

**Lemma 2.** Let $\beta_0$ be the critical value (the maximal value of the constant $\beta$, for which the Schrödinger operator with the potential $\beta V$ does not have a discrete spectrum). Then
\[
\inf \text{spec } \{-(1 - \gamma) \Delta + \beta_0 V(|x|)\} < -\delta \gamma^2,
\]
for some $\delta > 0$ and $\gamma$ small enough.

Moreover, there exists a function $f_\gamma(x)$, real, spherically symmetric and satisfying condition (61), with constants $C_{1,2}$ independent of $\gamma$, such that
\[
(1 - \gamma) \| \nabla f_\gamma \|^2 + \beta_0 (f_\gamma, V(|x|) f_\gamma) < -\delta \gamma^2 \| f_\gamma \|^2.
\]

**Proof.** Let us start by recalling some properties of the operator
\[
h_\beta = -\Delta + \beta V(x),
\]
$V(x) \leq 0$, radial, and compactly supported, with critical value $\beta = \beta_0$. For $\beta = \beta_0$ the operator $h_\beta$ has a so-called virtual level or zero-resonance. It means that the equation
\[
-\Delta \psi + \beta_0 V(x) \psi = 0
\]
has a generalized spherically symmetric solution $\tilde{\psi}$ with the following properties [VZ]:

(i) Let $B$ be a closure of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\| \psi \|_B = \| \nabla \psi \|$. Then $\tilde{\psi} \in B$. From this point we assume that $\tilde{\psi}$ is a normalized solution in the sense that $\| \tilde{\psi} \|_B = 1$. Notice, that $\tilde{\psi} \in L^2_{\text{loc}}(\mathbb{R}^3)$, but $\tilde{\psi} \not\in L^2(\mathbb{R}^3)$.

(ii) $-\Delta \tilde{\psi} \in L^2(\mathbb{R}^3)$ and $V(x) \tilde{\psi} \in L^2(\mathbb{R}^3)$.

(iii) Outside the support of $V(x)$ holds
\[
\tilde{\psi}(x) = C |x|^{-1}.
\]

The last property follows immediately from the fact that outside the support of $V(x)$ a radial solution of (74) can be written as $c_1 |x|^{-1} + c_2$, and $\psi \in B$ implies $c_2 = 0$.

Now we proceed directly to the proof Lemma 2. Let
\[
u \in C_0^\infty(\mathbb{R}^3), \quad u(x) \leq 1, \quad u(x) = 1 \text{ for } |x| \leq 1, \quad u(x) = 0 \text{ for } |x| \geq 2
\]
and set
\[
f_n(x) = \tilde{\psi}(x) u(|x| \gamma n^{-1}) \| \tilde{\psi}(x) u(|x| \gamma n^{-1}) \|^{-1}.
\]

Obviously $\| f_n(x) \| = 1$ and for large $n$

\[
\| \nabla f_n(x) \| \leq \| \tilde{\psi}(x) u(|x| \gamma n^{-1}) \|^{-1} \{ \| \nabla \tilde{\psi} \|_{|x| \leq 2\gamma n} \\
+ \| C |x|^{-1} \|_{\gamma^{-1} n \leq |x| \leq 2\gamma^{-1} n} \max \| \nabla u(|x| \gamma n^{-1}) \|) \}
\leq \| \tilde{\psi}(x) u(|x| \gamma n^{-1}) \|^{-1} \{ \| \nabla \tilde{\psi} \| + c(\gamma^{-1} n)^{1/2} \gamma n^{-1} \} \leq 2 \| \tilde{\psi}(x) u(|x| \gamma n^{-1}) \|^{-1}.
\]

(78)
Assume $V(x)$ is supported in a ball of radius $a_0$. Then $\tilde{\psi}(x) = C|x|^{-1}$ for $|x| \geq a_0$ and
\[
\|\tilde{\psi}(x)u(|x|\gamma n^{-1})\|^2 \geq 4\pi C^2 \int_{a_0 \leq |x| \leq 2\gamma^{-1}n} d|x|
= 4\pi C^2(2\gamma^{-1}n - a_0) \geq 4C^2\pi \gamma^{-1}n \quad (79)
\]
for $n \geq \frac{a_0}{\gamma}$.

The inequalities (78) and (79) imply the second relation in (61) with the constant $C_2$ independent of $n$ and $\gamma$. To check the first inequality in (61) let us estimate
\[
\|\Delta \tilde{\psi}(x)u(|x|\gamma n^{-1})\| \leq \|\Delta \tilde{\psi}\| + \left( \sum_{i=1}^{3} \int \left( \sum_{i=1}^{3} \frac{\partial \tilde{\psi}}{\partial x_i} \frac{\partial u}{\partial x_i} \right)^2 dx \right)^{\frac{1}{2}}
+ \left( \int \frac{C}{|x|^2} \Delta u(|x|\gamma n^{-1})dx \right)^{\frac{1}{2}}. \quad (80)
\]
According to (ii) the first term on the r.h.s. of (80) is bounded. The second term is also bounded, since $|\nabla u(|x|\gamma n^{-1})| \leq \text{const.}$ (recall that $\|\nabla \tilde{\psi}\| = 1$) and the last term is also bounded by a constant for $n \geq \frac{2a_0}{\gamma}$. Finally we arrive at
\[
\|\Delta \tilde{\psi}(x)u(|x|\gamma n^{-1})\| \leq C_1\|\nabla \tilde{\psi}(x)\|, \quad (81)
\]
which implies
\[
\|\Delta f_n(x)\| \leq C_1\|\nabla f_n(x)\|. \quad (82)
\]
To prove the Lemma it suffices now to show that for large $n$
\[
(1 - \gamma)\|\nabla f_n\|^2 + \beta_0(f_n, Vf_n) \leq -\delta \gamma^2\|f_n\|^2, \quad (83)
\]
with some $\delta > 0$ independent of $\gamma$. This is equivalent to
\[
(1 - \gamma)\|\nabla(\tilde{\psi}(x)u(|x|\gamma n^{-1}))\|^2 + \beta_0(\tilde{\psi}, V\tilde{\psi}) \leq -\delta \gamma^2\|\tilde{\psi}(x)u(|x|\gamma n^{-1})\|^2. \quad (84)
\]
Recall that
\[
\|\tilde{\psi}u(|x|\gamma n^{-1})\|^2 \leq c_3^4 \pi a_0^3 + 8\pi C^2 \int_{a_0}^{2\gamma^{-1}n} d|x| \leq 2c_4 \gamma^{-1}n, \quad (85)
\]
for large $n$, where $c_3 = \max_{|x| \leq a_0}|\tilde{\psi}(x)|$, and
\[
\|\nabla \tilde{\psi}\|^2 + \beta_0(\tilde{\psi}, V\tilde{\psi}) = 0, \quad (86)
\]
which implies
\[
(1 - \gamma)\|\nabla(\tilde{\psi}(x)u(|x|\gamma n^{-1}))\|^2 + \beta_0(\tilde{\psi}, V\tilde{\psi}) \leq
\leq -\gamma\|\nabla(\tilde{\psi})u\|\|\nabla u\|^2 + \|\nabla \tilde{\psi}\|^2 - \|\nabla(\tilde{\psi}u)\|^2 \quad (87)
\]
\[
\leq -\gamma \left[ \frac{1}{2}\|\nabla \tilde{\psi}\| - C\gamma^{1/2}n^{-1/2} \right]^2 + 3\|\nabla \tilde{\psi}\|^2|_{|x|\leq \gamma^{-1}n}
+ 2\|\tilde{\psi}\|\|\nabla u\|^2|_{\gamma^{-1}n \leq |x| \leq 2\gamma^{-1}n}. 
\]
For $n$ large we have
\[ i) \quad \| \nabla \bar{\psi} \|_{\gamma^{-1}n}^2 = 4\pi C^2 \int_{\gamma^{-1}n}^{\infty} |x|^{-2} d|x| = 4\pi C^2 \gamma n^{-1}, \]
\[ ii) \quad \| \bar{\psi} \|_{\gamma^{-1}n} \leq 2\gamma^{-1}n \leq C\gamma^2 n^{-2} \int_{\gamma^{-1}n}^{2\gamma^{-1}n} d|x| = C\gamma n^{-1}, \]
\[ iii) \quad C \gamma^{1/2} n^{-1/2} < \frac{1}{4} = \frac{1}{4} \| \nabla \bar{\psi} \|, \]
which implies together with (87)
\[- (1 - \gamma) \| \nabla (\bar{\psi}(x)u(|x| \gamma n^{-1})) \| ^2 + \beta_0(\bar{\psi}; V\bar{\psi}) \leq -\gamma/4 + 3C^2 \gamma n^{-1} + 2C\gamma n^{-1} \leq -\gamma/8 \leq -\frac{\gamma^2}{32C^2 \pi n} \| \bar{\psi} u(|x| \gamma n^{-1}) \| ^2. \]

To complete the proof of the Lemma it suffices now to choose $n$ so large that (90) holds true (notice, that it can be done uniformly in $\gamma$ for $\gamma \leq 1$) and for this $n$ take $\delta = \frac{1}{32} C^{-2} \pi^{-1} n^{-1}$, where $C$ is the constant in (79), which depends on the zero-resonance solution $\bar{\psi}$ only. \hfill \box

**Acknowledgment:** The work was partially supported by the European Union through its Training, Research, and Mobility program FMRX-CT 96-0001. C. Hainzl has been supported by a Marie Curie Fellowship of the European Community programme “Improving Human Research Potential and the Socio-economic Knowledge Base” under contract number HPMFCT-2000-00660. C. H. and S.-A. V. thank Robert Seiringer for many valuable comments.

**References**


Mathematisches Institut, LMU München, Theresienstrasse 39, 80333 Munich, Germany
E-mail address: hainzl@mathematik.uni-muenchen.de

Department of Mathematics, University of British Columbia, Vancouver, B.C. V6T 1Z2, Canada
E-mail address: vitali@math.ubc.ca

Mathematisches Institut, LMU München, Theresienstrasse 39, 80333 Munich, Germany
E-mail address: wugalter@mathematik.uni-muenchen.de