

# The Analogue of Bohm–Bell Processes on a Graph

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## Abstract

Bohm–Bell processes, of interest in the foundations of quantum field theory, form a class of Markov processes  $Q_t$  generalizing in a natural way both Bohm’s dynamical system in configuration space for nonrelativistic quantum mechanics and Bell’s jump process for lattice quantum field theories. They are such that at any time  $t$  the distribution of  $Q_t$  is  $|\psi_t|^2$  with  $\psi$  the wave function of quantum theory. We extend this class here by introducing the analogous Markov process for quantum mechanics on a graph (also called a network, i.e., a space consisting of line segments glued together at their ends). It is a piecewise deterministic process whose innovations occur only when it passes through a vertex.

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## 1 Introduction

We consider quantum mechanics on a *graph*  $\mathcal{G}$  (also called a *network*), i.e., on a topological and metric space consisting of one-dimensional manifolds glued together at their end points [20]. (It is not necessary for our purposes to regard the graph as embedded in  $\mathbb{R}^n$ .) We denote by  $\mathcal{G}$  the set of all points belonging to the graph: vertices and non-vertices together. The wave function at time  $t$  is a function  $\psi_t : \mathcal{G} \rightarrow \mathbb{C}$  on the graph (though one could also think of functions to  $\mathbb{C}^n$ ) and evolves according to the usual nonrelativistic Schrödinger equation

$$i\hbar \frac{\partial \psi_t}{\partial t} = -\frac{\hbar^2}{2} \Delta \psi_t + V \psi_t, \quad (1)$$

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understood in a suitable way (see Section 2). We introduce a Markov process  $(Q_t)_{t \in \mathbb{R}}$  in  $\mathcal{G}$  associated with a wave function  $\psi$  obeying (1).

This process is a contribution to the research program of providing for every quantum theory a canonical Markov process in its configuration space. Examples of such processes are: the motion of the configuration  $Q_t$  in Bohmian mechanics [6, 14, 18, 5] for nonrelativistic quantum mechanics, a dynamical system (and thus a deterministic process) in Euclidean space; Bell’s process [4, 15, 16, 8] for lattice quantum field theory, a Markovian pure jump process on a lattice; and the Markov processes employed in “Bell-type quantum field theories” [10, 11, 12, 13, 16], whose paths are piecewise Bohmian trajectories interrupted by stochastic jumps, and which can be regarded as a continuum analogue of Bell’s process, or as particle creation and annihilation added to Bohmian mechanics. To all of these processes we refer as “Bohm–Bell processes”; see [16] for an introduction. They are “guided” by the quantum wave function  $\psi$  in the sense that the transition probabilities are determined by  $\psi$ , and they are *equivariant processes* [12, 13, 14], i.e., such that at any time  $t$  the distribution of  $Q_t$  is  $|\psi_t|^2$ .

The investigation of Bell-type quantum field theories in [10, 11, 12, 13, 16, 8] has sparked interest in natural classes of processes generalizing both Bohmian mechanics and Bell’s process. One such class, not yet exactly defined but outlined in [13, Sec. 5.3], providing a canonical Markov process for a given Hamiltonian, configuration space, and wave function, is the class of “minimal processes.” I strongly expect that the process on the graph we are discussing is contained in this class, and in anticipation I call the process the “minimal graph process.”

What is novel about the minimal graph process is that graphs do not belong to the spaces on which such processes have been considered before, which are: Euclidean spaces [6], discrete coarse-grainings of Euclidean spaces [23], infinite-dimensional vector spaces (of field variables) [6], Riemannian manifolds [24, 9], lattices [4], countable unions of disjoint Euclidean spaces [10] or of disjoint Riemannian manifolds [13, 11, 12], and manifolds with boundaries [16].

The crucial difference between graphs and manifolds is that in a vertex of a graph three or more edges can meet, forming a Y-shaped (or, for more than three edges, +-shaped or \*-shaped) neighborhood of the vertex that is forbidden in a manifold. It is exactly these Y-shaped neighborhoods that are the source of a feature of the minimal graph process which is absent for the corresponding process associated with the Schrödinger equation (1) on a manifold: while the latter is deterministic, the minimal graph process is not, as it typically makes a random turn at every vertex, i.e., it selects at random one among the edges ending at this vertex, and moves away along that edge. The random turns at the vertices constitute, in fact, the only stochasticity in the process: once the process is on some edge, it moves deterministically, like the Bohmian motion on a manifold, until it arrives at a vertex, and the only random decision taken

there is along which edge to proceed. In particular, the paths of the minimal graph process are continuous. As we show in Section 3, a deterministic equivariant process with continuous paths is generically impossible on a graph; thus, the *topology enforces stochasticity*. An analogous connection between topology and stochasticity has been observed in [16, Sec. 6], in that case concerning *boundaries* of (rather than Y-shapes in) the configuration space.

The scope of this paper is modest. We define the minimal graph process, point out in what sense it is the unique analogue of Bohm–Bell processes, and compare it to Bohm–Bell processes and some of their limiting cases. We regard it as a natural mathematical extension of the class of “minimal processes” associated with quantum theories, beyond the realm in which it was considered so far. I think that the simplicity of the minimal graph process adds to the overall picture of naturalness of the “minimal processes”, and thus to the confidence with which one can propose physical theories employing minimal processes.

The paper is organized as follows. In Section 2 we define the minimal graph process. In Section 3 we show that the minimal graph process is uniquely selected by four postulates, and we discuss other equivariant processes on the graph. In Section 4 we compare the minimal graph process to the known jump processes. In Section 5 we show that the minimal graph process is a limiting case of Bell’s process. In Section 6 we discuss how it relates to a suitable limiting case of Bohmian mechanics. In Section 7 we consider the behavior of the process under symmetries. In Section 8 we put the connection between topology and stochasticity into perspective.

## 2 The Minimal Graph Process

If one tries to find an analogue of Bohmian mechanics on a graph, the simplest process one could choose is the minimal graph process. Let  $\mathcal{V}$  be the set of vertices of the graph  $\mathcal{G}$ ,  $\mathcal{E}$  the set of edges, and  $\mathcal{E}_q$  for  $q \in \mathcal{V}$  the set of edges ending at  $q$ . We assume (without loss of generality) that an edge cannot end at the same vertex on both sides; i.e., every closed path consists of at least two edges. Every edge is isometric to either an interval  $[0, a]$  of positive length or the half-infinite interval  $[0, \infty)$ . We assume that the graph is connected and has only finitely many vertices and edges.

The wave function  $\psi_t$  provides us with a probability density

$$\rho_t(q) = |\psi_t(q)|^2 \tag{2}$$

and a probability current vector

$$j_t(q) = \hbar \operatorname{Im}(\psi_t^*(q) \nabla \psi_t(q)), \tag{3}$$

at least at every non-vertex  $q \in \mathcal{G} \setminus \mathcal{V}$ . (In case that the value space of  $\psi_t$  is not  $\mathbb{C}$  but a higher-dimensional complex vector space  $\mathbb{C}^n$ , the product in the bracket in (3) should be

understood as an inner product in  $\mathbb{C}^n$ .) At a vertex  $q$ , there are several current vectors

$$j_{e,t}(q) = \hbar \operatorname{Im}(\psi_t^*(q) \nabla_e \psi_t(q)), \quad (4)$$

one for each edge  $e \in \mathcal{E}_q$  ending there; this corresponds to the fact that at  $q$  there is one (one-sided) derivative  $\nabla_e$  for each edge  $e \in \mathcal{E}_q$ . Together, the  $j_t(q)$  for  $q \in \mathcal{G} \setminus \mathcal{V}$  and the  $j_{e,t}(q)$  for  $q \in \mathcal{V}$  and  $e \in \mathcal{E}_q$  form what can be regarded as a *vector field*, denoted  $j_t$ , on  $\mathcal{G}$ , consisting of one element in each tangent space, where a vertex is thought of as having several tangent spaces, one for each edge.

The obvious choice for the law of motion along an edge (outside the vertices) is Bohm's [6, 14], i.e., the deterministic law

$$\frac{dQ_t}{dt} = v_t(Q_t) = \frac{j_t(Q_t)}{\rho_t(Q_t)}. \quad (5)$$

To have  $Q_t$  follow the vector field  $v_t = j_t/\rho_t$  ensures that the probability current  $\rho_t v_t$  of the process (at non-vertices) agrees with the prescribed current  $j_t$ , provided the process has distribution  $\rho_t$  as intended.

According to the probability distribution (2), the probability of  $Q_t \in \mathcal{V}$  vanishes, like for every other finite subset of  $\mathcal{G}$ ; this suggests that whenever  $Q_t$  reaches a vertex  $q$  it should leave  $q$  immediately, rather than spend some time sitting on  $q$ . Since we want that the probability flux of the process be given by  $j_t$  (and that the paths are continuous), we need that the flux into the vertex is as large as the flux out of the vertex, that is, that the net flux into the vertex is zero. This *Kirchhoff condition* can be expressed by the formula

$$\sum_{e \in \mathcal{E}_q} n_e(q) \cdot j_{e,t}(q) = 0, \quad (6)$$

where  $n_e(q)$  is the unit vector at  $q$  pointing in the direction of the edge  $e$  (that is, away from  $q$ ), and the dot  $\cdot$  denotes the inner product in the tangent space to the edge  $e$  (with  $e$  regarded as a Riemannian manifold with boundaries) at the point  $q$ . At a vertex at which just one single edge ends, (6) requires the current to vanish. The meaning of the Kirchhoff condition (6) is local conservation of probability at the vertex  $q$ ; no probability gets lost or added. Therefore it is the analogue, at the vertices, of the continuity equation

$$\frac{\partial \rho_t}{\partial t}(q) = -\nabla \cdot j_t(q), \quad (7)$$

which expresses the local conservation law at non-vertices  $q \in \mathcal{G} \setminus \mathcal{V}$ .

While (7), with (2) and (3) inserted, is a consequence of the Schrödinger equation (1), (6) is an additional requirement. The simplest way, and presumably the only practical way, of ensuring (6) for all times is to impose a boundary condition on the wave function

$\psi$  that implies (6).<sup>1</sup> Whereas (6) is a condition quadratic in  $\psi$ , the boundary condition on  $\psi$  should be linear: otherwise the acceptable wave functions would not form a linear space, and there would be little hope that the boundary condition could be conserved by the evolution of the wave function. A natural choice of boundary condition is thus a *Robin boundary condition*,

$$\alpha(q) \sum_{e \in \mathcal{E}_q} n_e(q) \cdot \nabla_e \psi(q) = \beta(q) \psi(q), \quad q \in \mathcal{V}, \quad (8)$$

where  $\alpha(q)$  and  $\beta(q)$  are real constants (and not both zero). Here, it is assumed that

$$\psi \text{ is continuous at vertices,} \quad (9)$$

so that  $\psi(q) \rightarrow \psi(q_0)$  as  $q \rightarrow q_0 \in \mathcal{V}$  along an edge. Together, (8) and (9) imply the Kirchhoff condition (6) on the current just as the Schrödinger equation (1) implies the continuity equation (7). This is seen, if  $\alpha(q) \neq 0$ , by multiplying (8) by  $\alpha(q)^{-1} \hbar \psi^*(q)$  and taking the imaginary part, observing that  $\alpha(q)$  and  $\beta(q)$  are real. In the case  $\alpha(q) = 0$ , (8) reduces to the *Dirichlet boundary condition*  $\psi(q) = 0$ , which also obviously implies the Kirchhoff condition (6) on the current. (The Dirichlet condition is the simplest condition at an external vertex (i.e., one with only one edge) though not very interesting for us at internal vertices as it excludes any flux of probability across the vertex.)

In fact, the Laplacian on the functions  $\psi$  (with the right degree of regularity, namely from the second Sobolev space on each edge) satisfying (8) and (9) is self-adjoint [26, 27, 19, 20]. Therefore, equations (1), (8), and (9) together define a unitary evolution on Hilbert space. The constants  $\alpha(q)$  and  $\beta(q)$  determine how much of an incoming wave gets reflected and how much transmitted, and with what phase shift.

(A remark, in brackets, on the other self-adjoint extensions of the Laplacian: On complex-valued functions, the extensions defined by (8) are, in fact, all self-adjoint extensions for continuous functions, i.e., assuming (9) [20, 19]. Further self-adjoint extensions exist if one drops (9); the most general local vertex condition defining a self-adjoint extension for complex-valued wave functions is the following [20, 19]: (i) Along each edge  $e \in \mathcal{E}_q$ , a limit of  $\psi$  in the vertex  $q$  exists,  $\psi(q, e) := \lim \psi(q')$  as  $q' \rightarrow q$  along  $e$ , though the limits may differ for different edges. (ii) Let  $F \in \mathbb{C}^d$ , where  $d = \#\mathcal{E}_q$  is the degree of the vertex  $q$ , be the vector with the components  $F_e = \psi(q, e)$ , and  $F' \in \mathbb{C}^d$  the vector with the components  $F'_e = n_e(q) \cdot \nabla_e \psi(q)$ . The further conditions are  $P^\perp F = 0$  and  $PF' + LPF = 0$ , where  $P$  is an orthogonal projection in  $\mathbb{C}^d$ ,  $P^\perp = 1 - P$  the

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<sup>1</sup>The term “boundary condition” is in a sense inappropriate, a sense in which “vertex condition” would be more appropriate: the condition concerns the behavior of  $\psi$  at vertices, and vertices are not (necessarily) boundaries of the graph. They are boundaries, however, of the edges glued to them, and that is how the similarity with boundary conditions in other quantum mechanical situations comes about.

complementary projection, and  $L$  a self-adjoint endomorphism on the range of  $P$ . This vertex condition includes (8) for  $P$  the projection on  $(1, 1, \dots, 1)$  and  $L = \alpha(q)^{-1}\beta(q)$  (in case  $\alpha(q) \neq 0$ ) respectively  $P = 0$  (in case  $\alpha(q) = 0$ ). This vertex condition also implies the Kirchhoff condition (6) on the flux, since the left hand side of (6) equals, up to a factor  $\hbar$ ,  $\text{Im}\langle F, F' \rangle = \text{Im}\langle PF, F' \rangle = \text{Im}\langle PF, PF' \rangle = -\text{Im}\langle PF, LPF \rangle = 0$ , where  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{C}^d$ . The Kirchhoff condition on the flux is all we need to be able to define the minimal graph process. It is only for the sake of simplicity that we restrict our attention to the simpler condition (8).)

Now that we have made precise the evolution of the wave function  $\psi_t$ , let us turn to the process  $Q_t$ . Once  $Q_t$  has reached a vertex  $q$  by its deterministic motion, a decision needs to be made about along which edge to leave the vertex. Those edges  $e$  for which  $n_e(q) \cdot j_{e,t}(q) < 0$  holds, do not possess trajectories—solutions of (5)—that begin at  $q$  at time  $t$ . Those edges  $e$ , in contrast, for which  $n_e(q) \cdot j_{e,t}(q) > 0$  holds, do. The borderline case  $n_e(q) \cdot j_{e,t}(q) = 0$  we can ignore. Thus, conditional on  $Q_t = q \in \mathcal{V}$ , the simplest way of choosing the edge  $e \in \mathcal{E}_q$  along which to leave  $q$  is to choose it at random with probability

$$\mathbb{P}_t(e|q) = \frac{[n_e(q) \cdot j_{e,t}(q)]^+}{\sum_{f \in \mathcal{E}_q} [n_f(q) \cdot j_{f,t}(q)]^+}, \quad (10)$$

where  $x^+ = \max(x, 0)$  denotes the positive part of  $x \in \mathbb{R}$ . Note that by construction  $\mathbb{P}_t(e|q) \geq 0$  and  $\sum_{e \in \mathcal{E}_q} \mathbb{P}_t(e|q) = 1$ .  $\mathbb{P}_t(e|q)$  is ill-defined when and only when the denominator vanishes (assuming that  $j_t$  is well defined), which happens, by (6), when and only when  $j_{e,t}(q) = 0$  for all  $e \in \mathcal{E}_q$ .

This completes the definition of the minimal graph process: the wave function  $\psi_t$  evolves according to the PDE (1) with the boundary conditions (8) and (9), and the process  $Q_t$  moves according to the ODE (5) with the stochastic law (10) at every vertex. We have left out of consideration the possibility of a topological phase factor associated with every non-contractible closed path (see [9] for a discussion), as this possibility does not affect those features of the process that we are interested in.

### 3 Equivariant Processes

The stochastic law (10), and thus the minimal graph process, is uniquely determined by the following requirements:

- (i)  $(Q_t)_{t \in \mathbb{R}}$  is a Markov process,
- (ii) it has continuous paths,
- (iii) it is equivariant, i.e.,  $|\psi_t|^2$  distributed at every  $t$ ,

(iv) the motion along the edges is Bohmian, i.e., given by (5).

To see this, consider a process satisfying (ii) and (iv). Such a process, whenever it is in a vertex  $q$ , waits a random time in  $q$ , which must almost surely be zero if it is equivariant, and then selects at random one of the edges for leaving  $q$ . If it is a Markov process, the probabilities for the various edges  $e \in \mathcal{E}_q$  do not depend on the past history of the process, not even on the edge along which it reached  $q$ ; thus, they are given by a function  $\mathbb{P}_t(e|q)$  of time  $t$  and vertex  $q$  alone. The probability flux (per time) out of  $q$  along  $e$  is then

$$J_{e,t}^{\text{out}}(q) = \mathbb{P}_t(e|q) \sum_{f \in \mathcal{E}_q} J_{f,t}^{\text{in}}(q), \quad (11)$$

where  $J_{f,t}^{\text{in}}(q)$  is the probability flux (per time) into  $q$  along  $f$ .

Now invoke the  $|\psi|^2$  distribution. The flux into  $q \in \mathcal{V}$  along  $e \in \mathcal{E}_q$  between  $t$  and  $t + dt$  equals the  $|\psi|^2$  measure of the Bohmian trajectories ending at  $q$  between  $t$  and  $t + dt$ .<sup>2</sup> It follows, since the arrival statistics of the Bohmian trajectories at  $q$  is given by the quantum current  $j_{e,t}(q)$  defined in (4), that

$$J_{e,t}^{\text{in}}(q) = j_{e,t}^{\text{in}}(q) := [n_e(q) \cdot j_{e,t}(q)]^-, \quad (12)$$

where  $x^- = \max(-x, 0)$  denotes the negative part of  $x \in \mathbb{R}$ . Similarly, to obtain the  $|\psi|^2$  distribution along the edge  $e$  at later times, it is necessary that the amount of probability leaving  $q$  per unit time along  $e$  is

$$J_{e,t}^{\text{out}}(q) = j_{e,t}^{\text{out}}(q) := [n_e(q) \cdot j_{e,t}(q)]^+. \quad (13)$$

That is because no other trajectories can contribute to the probability contents of any interval of  $e$  than those which started in the right time interval. Inserting (12) and (13) into (11), and observing that by the Kirchhoff condition (6) we have that

$$\sum_{e \in \mathcal{E}_q} j_{e,t}^{\text{in}}(q) = \sum_{e \in \mathcal{E}_q} j_{e,t}^{\text{out}}(q), \quad (14)$$

we obtain (10).

Note that while we required only equivariance, we obtained more, namely the “standard current property” [12, 13, 16]: not only the distribution density of the process

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<sup>2</sup>That means, in case all the trajectories along  $e$  ending at  $q$  during  $[t_1, t_2]$  existed already at time  $t_0 \leq t_1$  (rather than started, at one of the ends of  $e$ , after  $t_0$ ), that the flux into  $q$  along  $e$  during  $[t_1, t_2]$  is

$$\int_{t_1}^{t_2} J_{e,t}^{\text{in}}(q) dt = \int_e |\psi_{t_0}(q')|^2 1_{\{t_1 \leq \tau(q') \leq t_2\}} dq'$$

with  $\tau(q')$  the time of arrival at  $q$  (i.e.,  $\infty$  in case of no arrival) of the trajectory starting in  $q' \in e$  at time  $t_0$ .

$Q_t$  agrees with the quantum value (2), but also its probability current agrees with the quantum current as given by (3) and (4).

It is a corollary of this uniqueness result that generically, a process satisfying our requirements (i)–(iv) *cannot be deterministic*. (The exception is when the minimal graph process is deterministic, which is when it so happens that at every vertex at every time the outflux either vanishes or takes place along a single edge.)

In fact, already (ii) and (iii) alone are generically incompatible with determinism. To see this, let  $(Q_t)_{t \in \mathbb{R}}$  be a deterministic process with continuous paths on the graph consisting of three copies  $e_1, e_2, e_3$  of  $[0, \infty)$  joined at a single vertex  $q$ . We show that  $Q_t$  cannot be equivariant for wave functions  $\psi$  such that, during a time interval  $[t_1, t_2]$ ,  $n_{e_1}(q) \cdot j_{e_1,t}(q) < 0$  and  $n_{e_i}(q) \cdot j_{e_i,t}(q) > 0$  for  $i = 2, 3$ . The reason is essentially that, due to determinism, the flux into  $q$  along  $e_1$  leaves  $q$ , at every time, along either  $e_2$  or  $e_3$  but not both, though it would have to be split to maintain equivariance.

In detail, with the notation  $P_i(t) = \int_{e_i} |\psi_t(q')|^2 dq'$  for the  $|\psi|^2$  measure of  $e_i$ , we have that  $P_1$  is strictly decreasing since  $(d/dt)P_1 = n_{e_1}(q) \cdot j_{e_1,t}(q) < 0$  whereas  $P_2$  and  $P_3$  are strictly increasing,  $(d/dt)P_i > 0$ ,  $i = 2, 3$ . By continuity, a path  $t \mapsto Q_t$  can pass from one edge to another only by crossing  $q$ . Let  $e(t)$  and  $f(t)$  be the edges along which the path crossing  $q$  at time  $t$  reaches respectively leaves the vertex; this is well defined due to the assumed determinism. Let  $J_i(t) dt$  be the probability that the path  $s \mapsto Q_s$  crosses the vertex between  $t$  and  $t + dt$  and leaves along  $e_i$ . With the notation  $R_i(t) := \text{Prob}(Q_t \in e_i)$  for the probability contents of  $e_i$ , we have that for every subinterval  $[t_3, t_4] \subseteq [t_1, t_2]$ ,

$$R_i(t_4) \leq R_i(t_3) + \int_{t_3}^{t_4} J_i(t) dt. \quad (15)$$

Let  $S_i$  be the set of  $t \in [t_1, t_2]$  for which  $f(t) = e_i$ . Since during  $S_2$  no paths can enter  $e_3$ ,  $\int_{S_2} J_3(t) dt = 0$ . If  $Q_t$  were equivariant, then  $R_i(t) = P_i(t)$  for all  $t \in [t_1, t_2]$  and, by (15),  $(d/dt)P_i \leq J_i$ , and thus  $J_2(t) > 0$  and  $J_3(t) > 0$ . Therefore,  $S_2$  would have to be a null set, and thus  $R_2(t_2) \leq R_2(t_1) + \int_{S_2} J_2(t) dt = R_2(t_1) = P_2(t_1) < P_2(t_2)$ , in contradiction to equivariance.

Let us turn again to indeterministic processes. Other processes than the minimal graph process are possible when we drop the Markov property from our requirements. Then the distribution of the outgoing edge can depend on the past history of the process, and the most interesting possibility is perhaps that it is a function  $\mathbb{P}_t(e|q, f)$  of the edge  $f$  along which  $q$  was reached, yielding what could be called an almost-Markovian process. The condition on  $\mathbb{P}_t(e|q, f)$  deriving from (ii)–(iv) is

$$\sum_{f \in \mathcal{E}_q} [n_f(q) \cdot j_{f,t}(q)]^- \mathbb{P}_t(e|q, f) = [n_e(q) \cdot j_{e,t}(q)]^+, \quad (16)$$

together with

$$\sum_{e \in \mathcal{E}_q} \mathbb{P}_t(e|q, f) = 1, \quad (17)$$

and this is an underdetermined system of equations for the quantities  $\mathbb{P}_t(e|q, f)$  whenever there is influx along more than one edge and outflux along more than one edge.

Further equivariant processes are possible if we drop the requirement (iv) that the motion along the edges is Bohmian. One could consider instead an equivariant diffusion process such as Nelson’s *stochastic mechanics* [17, 21]. Diffusion processes can be defined on a graph as well [26, 27]; in order to specify such a process, one has to specify the diffusion constant and drift for every  $q \in \mathcal{G}$  and time  $t$ , and, in addition, for every vertex  $q$  and time  $t$  the probability distribution  $\mathbb{P}_t(e|q)$  on  $\mathcal{E}_q$  for which edge to select upon arrival at  $q$ .<sup>3</sup> With stochastic mechanics along the edges, the biggest difference for the uniqueness question is that the process can leave  $q$  along  $e$  even if  $n_e(q) \cdot j_{e,t}(q) < 0$ , and this opens up a lot of freedom.

## 4 Comparison with Bell-Type QFT

We now contrast the minimal graph process with Bell-type quantum field theories: these involve Markov processes on spaces  $\mathcal{Q}$  that are countable unions of disjoint manifolds (typically representing the configuration space of a variable number of particles), with stochastic jumps and deterministic, continuous (Bohmian) trajectories in between. The jumps, discontinuities in the path, occur at random times and lead to random destinations, with jump rates given by a formula in terms of  $\psi$  analogous to Bell’s [12]. Upon arrival at the destination, there is only one possibility, unlike at a graph vertex, in which direction the process can move on: it is defined by the Bohmian velocity vector field on  $\mathcal{Q}$  at the destination. Outside the context of quantum theory, similar Markov processes on the configuration space of a variable number of particles, representing many interacting particles with fragmentation and coagulation at random times, have been considered in statistical mechanics [3] and probability theory [22].

The stochastic jumps in a Bell-type quantum field theory correspond to the term  $H_I$  in the Hamiltonian  $H = H_0 + H_I$ , where  $H_0$ , a differential operator, is called the “free Hamiltonian” and  $H_I$ , an integral operator, the “interaction Hamiltonian”. In fact, this splitting of the Hamiltonian corresponds to a splitting  $L = L_0 + L_I$  of the generator  $L$  of the Markov process, where  $L_0$  generates the continuous motion and  $L_I$  the jumps [13] (for a general discussion of such a splitting of Markov generators see [3]), so that one could consider a process for  $H_0$  alone, which will be a process without jumps

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<sup>3</sup>The situation concerning the outgoing edge is more subtle though, as a diffusion process in one dimension returns to its starting point infinitely often within arbitrarily short times, so that one cannot speak of the “next edge” that the process enters after being in a vertex.

(in fact just the Bohmian motion). For the minimal graph process, in contrast, there does not exist a comparable splitting of the Hamiltonian in two contributions such that one would correspond to deterministic motion and the other to the stochastic decisions. There is, however, a correspondence we can make: the Hamiltonian is defined by a differential operator and a boundary condition; the differential operator corresponds to the deterministic Bohmian motion, while the boundary condition corresponds to the stochastic decision made at every vertex.

## 5 A Limiting Case of Bell's Process

It is well known [23, 25, 7] (though not on a rigorous level) that Bohmian mechanics is a limiting case of Bell's process: you approximate Euclidean space  $\mathbb{R}^n$  by a lattice  $\varepsilon\mathbb{Z}^n$  and the Laplacian by the lattice Laplacian, consider Bell's process and let  $\varepsilon \rightarrow 0$ . We now derive the minimal graph process as the limiting process of a suitable approximation by means of Bell's process. To this end, we replace each edge isometric to  $[0, a]$  by a lattice  $[0, a] \cap \varepsilon\mathbb{Z}$  and the Laplacian by the lattice Laplacian. Since we already know that along the edge, Bell's process converges to Bohmian mechanics, what remains to be investigated is the behavior of Bell's process at the vertices (which we include among the lattice sites). The rate for jumping from the vertex  $q$  to the nearest site along the edge  $e$ , which we denote symbolically by  $q + \varepsilon e$ , is [4, 15, 13]

$$\sigma_t(q + \varepsilon e|q) = \frac{\left[\frac{2}{\hbar}\text{Im} \psi_t^*(q + \varepsilon e) \langle q + \varepsilon e|H|q \rangle \psi_t(q)\right]^+}{\psi_t^*(q) \psi_t(q)}. \quad (18)$$

Since  $\langle q + \varepsilon e|H|q \rangle = \hbar^2/2\varepsilon^2$  for the lattice Laplacian, and since  $\psi(q + \varepsilon e) - \psi(q)$  is of the order  $\varepsilon$ , the jump rate is of the order  $\varepsilon^{-1}$ ; thus, the waiting time is of the order  $\varepsilon$  and goes to zero in the continuum limit  $\varepsilon \rightarrow 0$ . We are interested in the distribution over the edges. The probability that the process leaves  $q$  along  $e$  is

$$\frac{\sigma_t(q + \varepsilon e|q)}{\sum_{f \in \mathcal{E}_q} \sigma_t(q + \varepsilon f|q)} = \frac{\left[\frac{2}{\hbar}\text{Im} \psi_t^*(q + \varepsilon e) \langle q + \varepsilon e|H|q \rangle \psi_t(q)\right]^+}{\sum_{f \in \mathcal{E}_q} \left[\frac{2}{\hbar}\text{Im} \psi_t^*(q + \varepsilon f) \langle q + \varepsilon f|H|q \rangle \psi_t(q)\right]^+} = \frac{[n_e(q) \cdot j_{e,t}(q)]^+}{\sum_{f \in \mathcal{E}_q} [n_f(q) \cdot j_{f,t}(q)]^+},$$

where  $t$  is the time at which the process leaves  $q$ , and  $n_e(q) \cdot j_{e,t}(q)$  denotes the probability flux out of  $q$  into  $e$ , which in the lattice model equals  $\frac{2}{\hbar}\text{Im} \psi_t^*(q + \varepsilon e) \langle q + \varepsilon e|H|q \rangle \psi_t(q)$ . This obviously converges to (10), as we claimed.

## 6 Sort of Limiting Case of Bohmian Motion

Consider a graph  $\mathcal{G}$  isometrically embedded in  $\mathbb{R}^n$  and Bohmian mechanics in  $\mathbb{R}^n$  with a potential  $V$  that forces the particle to stay  $\varepsilon$ -close to  $\mathcal{G}$ . As we take the limit  $\varepsilon \rightarrow 0$ ,

does the process converge to the minimal graph process? The answer is in general no, but it seems plausible that the Markovization of the limiting process is the minimal graph process. The following example illustrates what happens.

Take  $n = 2$  and  $\mathcal{G}$  consisting of the four half-axes with the origin as the only vertex. An example potential that keeps  $Q_t$  close to  $\mathcal{G}$  is  $V(x, y) = \min\{(x/\varepsilon)^2, (y/\varepsilon)^2\}$ . Every point  $q$  in the plane minus the two diagonals possesses a unique closest point  $\pi(q)$  on  $\mathcal{G}$ : indeed, taking out the two diagonals decomposes the plane into four quadrants, each containing one half-axis, and  $\pi$  on a quadrant is the orthogonal projection to that half-axis. If  $Q_t^\varepsilon$  is the Bohmian path,  $\pi(Q_t^\varepsilon)$  is a process on  $\mathcal{G}$ , and one could imagine that for suitable choice of the initial wave function  $\psi_0^\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{C}$  as a function of  $\varepsilon$ ,  $\pi(Q_t^\varepsilon)$  possesses a limiting process  $Q_t^0$  as  $\varepsilon \rightarrow 0$ . One could also imagine that the discontinuity that occurs in  $t \mapsto \pi(Q_t^\varepsilon)$  whenever  $Q_t^\varepsilon$  crosses a diagonal vanishes in the limit, as  $Q_t^\varepsilon$  crosses the diagonal in an  $\varepsilon$ -neighborhood of the origin.

However, the transversal coordinate of  $Q_t^\varepsilon$ , the one that is projected out by  $\pi$ , may decide about the edge along which to leave the central region, as depicted in Figure 1. As a consequence, the probability distribution for the outgoing edge may depend on the ingoing edge, so that the limiting process  $Q_t^0$  is not Markovian, but instead the kind of almost-Markovian process described by (16) and (17).

Still, a process of this kind has the property that its Markovization is the minimal graph process (for the same wave function). The *Markovization* of a stochastic process  $Q_t$  is defined as the Markov process  $\tilde{Q}_t$  with

$$\text{Prob}(\tilde{Q}_t \in B \text{ and } \tilde{Q}_{t+dt} \in C) = \text{Prob}(Q_t \in B \text{ and } Q_{t+dt} \in C), \quad (19)$$

for all sets  $B, C$  and all  $t$ . It is not obvious that a Markovization exists. In contrast, in discrete time the Markovization, with  $dt$  replaced by the time step, obviously exists and is unique in law. When  $\tilde{Q}_t$  exists, it has the same one-time marginals and the same transition probabilities (not conditional on the prior history) for infinitesimal time differences  $dt$ .

For an almost-Markovian process moving with velocities  $v_t$  along the edges and selecting the outgoing edge  $e$  at a vertex  $q$  reached along  $f$  with distribution  $\mathbb{P}_t(e|q, f)$ , the Markovization looks as follows: it moves again with velocities  $v_t$  along the edges and selects the outgoing edge at a vertex  $q$  with distribution  $\tilde{\mathbb{P}}_t(e|q)$  given by

$$\tilde{\mathbb{P}}_t(e|q) = \frac{\sum_{f \in \mathcal{E}_q} \mathbb{P}_t(e|q, f) \rho_{f,t}(q) [n_f(q) \cdot v_{f,t}(q)]^-}{\sum_{f \in \mathcal{E}_q} \rho_{f,t}(q) [n_f(q) \cdot v_{f,t}(q)]^-}. \quad (20)$$

If the process is equivariant,  $v_t$  is the Bohmian velocity vector field, and  $\mathbb{P}_t(e|q, f)$  satisfies the equivariance condition (16), then, by the Kirchhoff condition (6), the distribution of the edges  $\tilde{\mathbb{P}}_t(e|q)$  of the Markovization equals the one of the minimal graph process, (10).

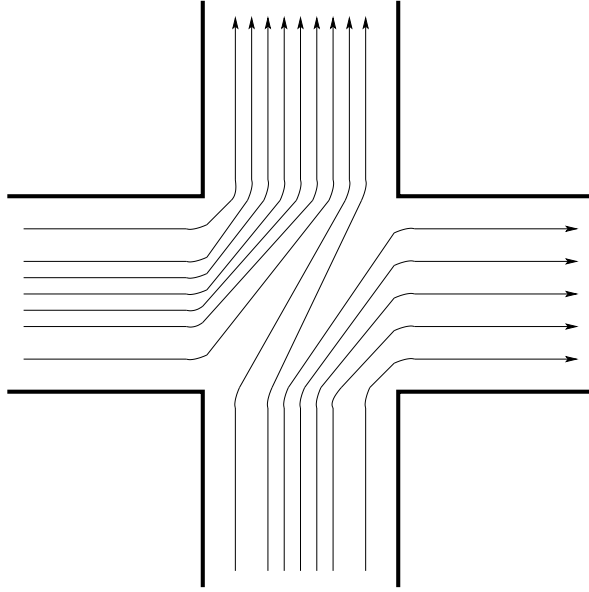


Figure 1: The Bohmian trajectories, qualitatively, in an example case in which the motion is confined to an  $\varepsilon$ -neighborhood of the axes (bounded by the bold lines). In the limit  $\varepsilon \rightarrow 0$ , the motion takes place along the axes, and may turn from one axis to the other at the origin. In this example, trajectories coming in from the left go out upwards whereas trajectories coming in from below go out either upwards or to the right; therefore, the projection  $\pi(Q_t)$  to the axes is not Markovian.

## 7 Symmetries

The minimal graph process respects the symmetries of the Schrödinger equation. That is, suppose that  $\mathcal{G}$  possesses an isometry  $\varphi : \mathcal{G} \rightarrow \mathcal{G}$ . Then  $\varphi(Q_t)$  is again a minimal graph process, associated with the wave function  $\psi \circ \varphi$ , which obeys the Schrödinger evolution with potential  $V \circ \varphi$ . If  $V$  is symmetric under  $\varphi$ , one obtains in this way further solutions  $\psi_t, Q_t$  of the same set of defining equations (1), (5), and (10). If, in addition,  $\psi_t$  is symmetric under  $\varphi$ , it remains so for all times, and the distribution of the process  $(Q_t)_{t \in \mathbb{R}}$ , regarded as a measure on the path space, is symmetric under the action of  $\varphi$ . Note that the isometries of  $\mathcal{G}$  always form a finite set (of at most  $k!2^k$  elements if the graph has  $k = \#\mathcal{E}$  edges, since every isometry defines a permutation of the edges, and there are only two ways of isometrically mapping one edge to another one of the same length).

Another symmetry the minimal graph process respects is time reversal:  $(Q_{-t})_{t \in \mathbb{R}}$  is again a minimal graph process, associated with the wave function  $\psi'_t = \psi_{-t}^*$ . (As usual in quantum mechanics, the wave function has to be replaced, under time reversal, by its complex conjugate. Then  $\psi'$  solves the Schrödinger equation again.) To see this,

note that, as a consequence of the conjugation, the currents (3) change sign while the density  $|\psi|^2$  remains unchanged. Therefore, the Bohmian velocities (5) change sign, as they should. If the process is in a vertex  $q$  at time  $t$ , then the probability that it came along  $f$  and leaves along  $e$  is, by (10), proportional to  $[n_f(q) \cdot j_{f,t}(q)]^- [n_e(q) \cdot j_{e,t}(q)]^+$ . Therefore, (10) holds again for the reversed process with the reversed currents.

## 8 Topology and Stochasticity

We can regard graphs as test cases, or toy models, for the more complicated spaces of higher dimension arising from gluing together parts of Euclidean spaces or manifolds. Such spaces are not at all eccentric as configuration spaces; especially for configuration spaces for a variable number of particles, is it a natural thought to identify certain configurations, and thus to glue together different parts of the configuration space, initially given as disjoint manifolds. For example, one may think of identifying the configuration  $q$  consisting of an electron at  $\mathbf{x} \in \mathbb{R}^3$  and a photon at the same location  $\mathbf{x}$  with the configuration  $q'$  consisting of just an electron at  $\mathbf{x}$ . In this way, the act of absorption no longer corresponds to a discontinuity in configuration space. As another, though similar, example, one may identify the configuration  $q$  consisting of a particle at  $\mathbf{x}$  and an anti-particle at the same location  $\mathbf{x}$  with the vacuum configuration. Such glued configuration spaces have been considered in [1, 2] for a study of the spin–statistics connection.

Taking graphs as test cases, a feature we observe is that the special topological situation we encounter at vertices (shapes like  $\Upsilon$ ,  $+$ ,  $*$  etc.) inevitably leads to stochasticity. This could be connected to the stochasticity of Bell-type quantum field theories, which is associated with the annihilation and even more with the creation of particles. After all, creation and annihilation events involve crossing from one sector of configuration space to another (corresponding to a different particle number), and if, as we suggested above, the sectors are glued together, then one should expect stochasticity exactly at the annihilation and creation events.

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